Transformation from VMEC to Boozer Coordinates

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In this memo, the transformation equation between Boozer magnetic coordinates[1] and VMEC coordinates[2] is derived. (Note: for clarity, here we use the angle argument convention $m\theta_V + n\phi_V$ which is *opposite* the usual toroidal angle convention in the VMEC code, that is, *n*->-*n*.) Both coordinate systems use the toroidal flux function ψ as the radial variable. In Boozer coordinates, the contravariant and covariant components of the magnetic field define the angular variables θ_B and ϕ_B (the poloidal angle and the toroidal angle, respectively), through the relations:

$$\begin{aligned} B &= \nabla\psi \times \nabla\theta_B + \iota \nabla\phi_B \times \nabla\psi \\ &= g\nabla\phi_B + I\nabla\theta_B + \nu\nabla\psi \end{aligned} \tag{1}$$

Here, $\iota(\psi)$ is the rotational transform, $g(\psi)$ is the poloidal current (or toroidal magnetic field) function, $I(\psi)$ is the toroidal current function, and $\nu(\psi,\theta_B,\phi_B)$ is a periodic function representing the Pfirsch-Schlüter current. Alternatively, the contravariant representation of the magnetic field in VMEC coordinates can be written:

$$B = \nabla\psi \times \nabla\theta_V^* + \iota \nabla\phi_V \times \nabla\psi \tag{2}$$

Here, $\theta_V^* = \theta_V + \lambda$ is the VMEC poloidal angle for which the magnetic field lines are straight, θ_V is the poloidal angle used in the VMEC code, and $\phi_V = \phi$ is the cylindrical toroidal angle. The stream function $\lambda(\psi,\theta_V,\phi_V)$ is introduced in VMEC to accelerate the convergence in Fourier space for the inverse representation of *R* and *Z*, and its Fourier spectrum is available as standard output from VMEC on the half-integer radial mesh. [(VMEC uses a normalized radial flux coordinate *s*, so that the toroidal flux is $\psi = \psi(s)$].

The transformation between the Boozer and VMEC angles is constrained by toroidal and poloidal periodicity and the requirement that the straight field line (contravariant) form of **B** be preserved. This results in the following relations:

$$\begin{aligned}\theta_B &= \theta_V + \lambda(\psi,\theta_V,\phi_V) + \iota p(\psi,\theta_V,\phi_V) \\ \phi_B &= \phi_V + p(\psi,\theta_V,\phi_V)\end{aligned} \tag{3}$$

Here, p is a doubly-periodic transformation function that will be determined. A field-line differential equation for p is derived in the standard way by considering the transformation of the Jacobian $J \equiv (\nabla\varphi \times \nabla\theta \cdot \nabla\phi)^{-1}$ between the two representations and cylindrical coordinates. The result, upon using Eq. (3) to express J_B in terms of J_V, is

$$\frac{1}{J_B} = \frac{1}{J_V} + B \cdot \nabla p \tag{4}$$

The Jacobian in VMEC coordinates, J_V, is calculated at half-integer radial mesh points from standard VMEC equilibrium output. The Boozer Jacobian J_B is related to the magnetic field strength by taking the dot product of the co- and contra-variant forms for **B** in Eq. (1), so that Eq. (4) becomes:

$$B \cdot \nabla p = \left(\frac{B^2}{g + \iota I} \right) - \frac{1}{J_V} \ . \tag{5}$$

This can be solved algebraically for p (on irrational surfaces) in Fourier space in terms of the VMEC spectra of $J_V B^2$:

$$p_{mn} = \frac{-i}{\iota m + n} \left[\frac{(J_V B^2)_{mn}}{g + \iota I} \right] \ . \tag{6}$$

Here, $i=\sqrt{-1}$. Eq. (6) applies to all but the $(m,n)=(0,0)$ mode of p, which can be chosen equal to zero. The solubility condition for Eq. (5) requires that $\left(J_V B^2\right)_{00} = g + iI$.

The current flux functions g and I can be determined from the VMEC output in terms of the surface-averaged covariant components of **B**. Indeed, this can be readily seen by inserting the transformation Eq. (3) into the covariant Boozer representation for **B**:

$$B = g\nabla\phi_V + I\left(\nabla\theta_V + \nabla\lambda\right) + \left(g + \iota I\right)\nabla p + \nu'\nabla\psi \ . \tag{7}$$

Here, $\nu' = \nu + Ip\left(d\iota/d\psi\right)$. The covariant components of the magnetic field, in VMEC coordinates, are obtained from Eq. (7):

$$\begin{aligned} B_{\theta_V} &\equiv J_V \nabla\phi_V \times \nabla\psi \cdot B = I\left(1+\lambda_{\theta V}\right) + \left(g+\iota I\right)p_{\theta_V} \\ B_{\phi_V} &\equiv J_V \nabla\psi \times \nabla\theta_V \cdot B = g + I\lambda_{\phi_V} + \left(g+\iota I\right)p_{\phi_V} \end{aligned} \tag{8}$$

In Eq. (8), $\left(p_\alpha, \lambda_\alpha\right) = \left(\partial p/\partial\alpha, \partial\lambda/\partial\alpha\right)$ for $\alpha = \left(\theta_V, \phi_V\right)$.

Note that the quantities on the left of Eq. (8) can be calculated in VMEC coordinates from VMEC output. Integrating Eq. (8) over both angles to annihilate the (unknown) transformation function p yields solubility constraints determining the two current flux functions:

$$\begin{aligned} I &= \overline{B}_{\theta_V} \\ g &= \overline{B}_{\phi_V} \end{aligned} \tag{9}$$

The overbar in Eq. (9) denotes the angle average $\left(2\pi\right)^{-2}\iint d\theta_V d\phi_V$ over the VMEC angles. Note that the two relations in Eq. (8) are indeed consistent, since $J\cdot\nabla\psi \equiv 0 \propto \partial B_{\theta_V}/\partial\phi_V - \partial B_{\phi_V}/\partial\theta_V$.

Equation (8) provides a nonsingular alternative to Eq. (5) for determining the transformation function p. It corresponds to a factorization of B^2 into co- and contra-variant components and eliminates the singular denominator occurring at the rational

surfaces in Eq. (6). In Fourier space, Eq. (8) can be solved for the spectrum of the transformation function p:

$$\begin{aligned} I\lambda_{mn} + (g + \iota I)\, p_{mn} &= \frac{(B_{\theta V})_{mn}}{im} \text{ for } m \neq 0 \\ &= \frac{(B_{\phi V})_{mn}}{in} \text{ for } n \neq 0 \end{aligned} \tag{10}$$

To demonstrate the application of these transformation equations, consider the problem of converting any scalar function Ω from VMEC to Boozer coordinates. (A scalar satisfies $\Omega(\theta_V, \phi_V) = \Omega\left[\theta_B(\theta_V, \phi_V), \phi_B(\theta_V, \phi_V)\right]$). For example, $\Omega \in \left(R, Z, p, B^2\right)$ comprises a useful set of such invariants. From (R,Z), one can construct the metric tensor g_{ij} and, subsequently, all components of the magnetic field. Converting p from VMEC to Boozer coordinates allows one to reverse the transformation in Eq. (3). This provides a useful method for checking the numerical accuracy of the transformation scheme. To obtain the Boozer spectrum of Ω (assuming its VMEC coordinate representation is known), the transformation Eq. (3) can be applied as follows:

$$\begin{aligned} \Omega^B_{mn} &\equiv \frac{1}{(2\pi)^2} \iint d\theta_B d\phi_B e^{-i(m\theta_B + n\phi_B)} \Omega(\theta_B, \phi_B) \\ &= \frac{1}{(2\pi)^2} \iint d\theta_V d\phi_V e \frac{\partial(\theta_B, \phi_B)}{\partial(\theta_V, \phi_V)} \Omega(\theta_V, \phi_V) \times \\ &\qquad e^{-i\left[m(\theta_V + \lambda + \iota p) + n(\phi_V + p)\right]} \end{aligned} \tag{11}$$

The Jacobian between Boozer and VMEC coordinates is obtained from Eq. (3):

$$\begin{aligned} \frac{\partial(\theta_B, \phi_B)}{\partial(\theta_V, \phi_V)} &= (1 + \lambda_{\theta_V})(1 + p_{\phi_V}) + p_{\theta_V}(\iota - \lambda_{\phi_V}) \\ &= [B^{\mathrm{v}}(1 + p_{\phi_V}) + B^{u} p_{\theta_V}] / \Phi' \end{aligned} \tag{12}$$

Note that the Jacobian is area-preserving (as it must be).

References

1. A. H. Boozer, Phys. Fluids **24**, 2002 (1981).

2. S. P. Hirshman, W. I. van Rij, P. Merkel, Comp. Phys. Comm. **43**, 143 (1986).